4.6 Fundamental Theorem of Calculus Part 1

1. C $\dfrac{d}{dx}\displaystyle\int_1^{x^2}\sqrt{3+t^2}\,dt=\sqrt{3+(x^2)^2}\cdot\dfrac{d}{dx}(x^2)=2x\sqrt{3+x^4}$

2. C $F'(x)=\dfrac{d}{dx}\displaystyle\int_0^{\sin x}\dfrac{dt}{\sqrt{1-t^2}}=\dfrac{1}{\sqrt{1-\sin^2 x}}\cdot\dfrac{d}{dx}(\sin x)=\dfrac{\cos x}{\sqrt{\cos^2 x}}=\dfrac{\cos x}{\cos x}=1$

3. B $F'(x)=\dfrac{d}{dx}\displaystyle\int_0^{\sqrt{x}}\cos(t^2)\,dt=\cos(\sqrt{x})^2\cdot\dfrac{d}{dx}(\sqrt{x})=\cos x\cdot\dfrac{1}{2\sqrt{x}}$

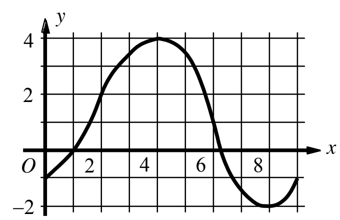
$F'(4)=\cos 4\cdot\dfrac{1}{2\sqrt{4}}=\dfrac{\cos 4}{4}$

4. C $f'(x)=\dfrac{d}{dx}\displaystyle\int_0^{x}\cos(t^2+2)\,dt=\cos(x^2+2)$. Use a graphing calculator to find the interval on which f' is positive. f' is positive on $[1.647, 2.419]$, thus f is increasing on that interval.

5. B $f'(x)=\dfrac{d}{dx}\displaystyle\int_0^{\sqrt{x}}g(t)\,dt=g(\sqrt{x})\cdot\dfrac{d}{dx}(\sqrt{x})=g(\sqrt{x})\cdot\dfrac{1}{2\sqrt{x}}$

$f'(4)=g(\sqrt{4})\cdot\dfrac{1}{2\sqrt{4}}=g(2)\cdot\dfrac{1}{4}$. The graph shows $g(2)=2$.

So, $f'(4)=2\cdot\dfrac{1}{4}=\dfrac{1}{2}$



graph of g

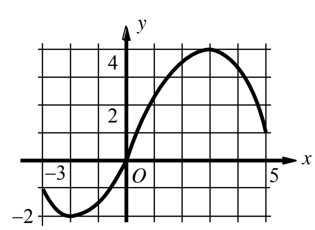
6. A $F'(x)=\dfrac{d}{dx}\displaystyle\int_0^{x^2}\dfrac{\sqrt{t^2+3}}{2t}\,dt=\dfrac{\sqrt{(x^2)^2+3}}{2(x^2)}\cdot\dfrac{d}{dx}(x^2)=\dfrac{\sqrt{x^4+3}}{2x^2}(2x)=\dfrac{\sqrt{x^4+3}}{x}$

$F''(x)=\dfrac{x(4x^3/2\sqrt{x^4+3})-\sqrt{x^4+3}}{x^2}$

$\Rightarrow\ F''(1)=\dfrac{1(4/2\sqrt{1^4+3})-\sqrt{1^4+3}}{1^2}=\dfrac{1-2}{1}=-1$

7. (a) $-3\le 2x-1\le 5\ \Rightarrow\ -2\le 2x\le 6\ \Rightarrow\ -1\le x\le 3$

The domain of g is $-1\le x\le 3$.



graph of f

(b) $g'(x)=\dfrac{d}{dx}\displaystyle\int_{-3}^{2x-1}f(t)\,dt=f(2x-1)\cdot\dfrac{d}{dx}(2x-1)=2f(2x-1)$

$g'(3)=2f(5)=2\cdot 1=2$

(c) $g'=f$ is negative for $x<0$ and $g'=f$ is positive for $x>0$, so the maximum is at an endpoint.

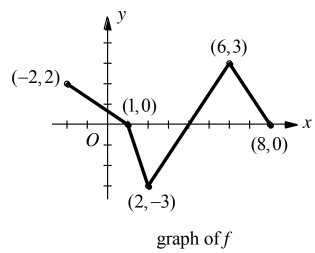
$g(-1)=\displaystyle\int_{-3}^{-3}f(t)\,dt=0$

$g(3)=\displaystyle\int_{-3}^{5}f(t)\,dt>0$, since the area above the x-axis is greater than the area below the x-axis.

Therefore g is at a maximum when $x=3$.

8. (a) $g(x)=\int_{-2}^{x} f(t)\,dt$ $\quad g'(x)=\dfrac{d}{dx}\int_{-2}^{x} f(t)\,dt=f(x)$

$g'(1)=f(1)=0$



graph of f

(b) The points of inflection are $x=2$ and $x=6$, because
$g'=f$ changes from decreasing to increasing at $x=2$ and
$g'=f$ changes from increasing to decreasing at $x=6$.

(c) Average rate of change $=\dfrac{g(8)-g(2)}{8-2}=\dfrac{1}{6}\int_{2}^{8} f(t)\,dt=\dfrac{1}{6}\left[-\dfrac{1}{2}(2)(3)+\dfrac{1}{2}(4)(3)\right]=\dfrac{1}{2}$

(d) There are two values of c . Since there are two points between 4 and 8 where $f(c)=\dfrac{1}{2}$.

4.7 Fundamental Theorem of Calculus Part 2

1. B $\quad f(3)-f(1)=\int_{1}^{3} \dfrac{\sqrt{x}}{1+x^3}\,dx$

Use a graphing calculator to find the value of $\int_{1}^{3} \dfrac{\sqrt{x}}{1+x^3}\,dx$.

$\boxed{y_1=\sqrt{x}/(1+x^3)}$, $\boxed{\text{2nd}}$, $\boxed{\text{CALC}}$, $\boxed{7:\int f(x)\,dx}$, Lower Limit? $\boxed{x=1}$, Upper Limit? $\boxed{x=3}$.

$\int_{1}^{3} \dfrac{\sqrt{x}}{1+x^3}\,dx=0.397$

$f(3)=f(1)+0.397=2.397$

2. C $\quad f(5)-f(-1)=\int_{-1}^{5} \cos(x^2-1)\,dx=1.524$

Use a graphing calculator to find the value of $\int_{-1}^{5} \cos(x^2-1)\,dx$.

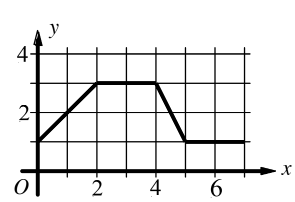
$f(5)=f(-1)+1.524=1.5+1.524=3.024$ $\quad f(-1)=1.5$

3. A $\quad g(3)-g(0)=\int_{0}^{3} \sqrt{x^4-3x+4}\,dx=9.966$ $\quad$ Use a graphing calculator.

$g(0)=g(3)-9.966=7-9.996=-2.996$ $\quad g(3)=7$

4. E $\quad \int_{2}^{10} f(\tfrac{1}{2}x)\,dx=\left[2F(\tfrac{1}{2}x)\right]_{2}^{10}=2\left[F(5)-F(1)\right]$

5. D $\quad F(6)-F(1)=\int_{1}^{6} f(x)\,dx=2.5+6+2+1=11.5$



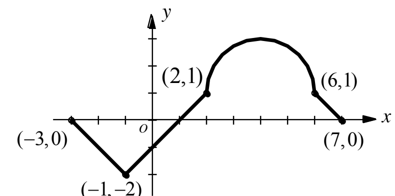
6. (a) $f(2)-f(-3)=\int_{-3}^{2} f'(t)\,dt=-\frac{1}{2}(4)(2)+\frac{1}{2}(1)(1)=-\frac{7}{2}$

$f(-3)=f(2)+\frac{7}{2}=3+\frac{7}{2}=\frac{13}{2}$

$f(7)-f(2)=\int_{2}^{7} f'(t)\,dt=(1)(4)+\frac{1}{2}(\pi\cdot 2^2)+\frac{1}{2}(1)(1)$

$=\frac{9}{2}+2\pi$

$f(7)=f(2)+\frac{9}{2}+2\pi=\frac{15}{2}+2\pi$



graph of f'

(b) The graph shows $f'(2)=1$. The tangent line at $(2,3)$ is $y-3=1(x-2) \Rightarrow y=x+1$.

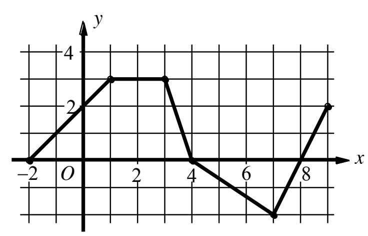
(c) f is increasing on $(1,7)$ since $f'>0$ for $1<x<7$.

(d) The graph of f is concave up on $-1<x<4$ because $f'(x)$ is increasing on this interval, $f''(x)$ must be positive on this interval.

7. (a) $g(0)=\int_{-2}^{0} f(t)\,dt=\frac{1}{2}(2)(2)=2$

$g'(0)=f(0)=2$

$g''(0)=f'(0)=1$ Slope of f at $x=0$ is 1.



graph of f

(b) The graph of f is concave up on $-2<x<1$ and $7<x<9$ because $g''(x)=f'(x)$ is increasing on these intervals.

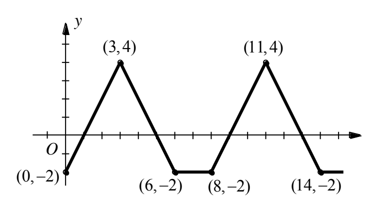
(c) $g(x)$ is increasing on $(-2,4)$ and $(8,9)$ because $g'(x)=f(x)>0$.

8. (a) $h(x)=\int_{0}^{x} f(t)\,dt$

$h(8)=\int_{0}^{8} f(t)\,dt=-\frac{1}{2}(1)(2)+\frac{1}{2}(4)(4)-\frac{1}{2}(1)(2)-(2)(2)=2$

$h'(6)=f(6)=-2$

$h''(4)=f'(4)=\frac{4+2}{3-6}=-2$



graph of f

(b) h has a relative minimum at $x=1$ because $h'=f$ changes from negative to positive at $x=1$.
h has a relative maximum at $x=5$ because $h'=f$ changes from positive to negative at $x=5$.

(c) The function value of h increase by 2 for every increase of 8 in x.

$h(35)=\int_{0}^{35} f(t)\,dt=\int_{0}^{32} f(t)\,dt+\int_{32}^{35} f(t)\,dt=4\int_{0}^{8} f(t)\,dt+\int_{0}^{3} f(t)\,dt$

$=4\cdot 2-\frac{1}{2}(1)(2)+\frac{1}{2}(2)(4)=11$

$h'(35)=f(35)=f(3)=4$

An equation for the line tangent to the graph of h at $x=35$ is
$y-11=4(x-35)$

4.8 Integration by Substitution

1. D Let $u = x^{3/2}$, then $du = \frac{3}{2}x^{1/2}dx$.

$$\int \sqrt{x}\sin(x^{3/2})\,dx = \int \frac{2}{3}\sin(u)\,du = -\frac{2}{3}\cos u + C = -\frac{2}{3}\cos(x^{3/2}) + C$$

2. D If $u = 2 - x$, then $du = -dx$ and $x = 2 - u$. When $x = 1$, $u = 1$ and when $x = 3$, $u = -1$.

$$\int_1^3 x\sqrt{2-x}\,dx = \int_1^{-1} (2-u)\sqrt{u}\,du = \int_{-1}^1 (u-2)\sqrt{u}\,du$$

3. C Let $u = x + k$, then $du = dx$. When $x = -1$, $u = k - 1$ and when $x = 3$, $u = k + 3$.

$$8 = \int_{-1}^3 f(x+k)\,dx = \overbrace{\int_{k-1}^{k+3} f(u)\,du = \int_{k-1}^{k+3} f(x)\,dx}^{\text{Integrals do not depend on the variable that is used.}}$$

4. A Let $u = 6 - x$, then $du = -dx$. When $x = 0$, $u = 6$ and when $x = 6$, $u = 0$.

$$\int_0^6 f(6-x)\,dx = \int_6^0 -f(u)\,du = \int_0^6 f(u)\,du = \int_0^6 f(x)\,dx = 12$$

5. B If $u = 1 + \sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}dx \Rightarrow \frac{1}{\sqrt{x}}dx = 2du$.

$$\int \frac{(1+\sqrt{x})^{3/2}}{\sqrt{x}}\,dx = \int (u)^{3/2}2du = 2\int u^{3/2}du$$

6. E If $u = \sec\theta$, then $du = \sec\theta\tan\theta d\theta$, so $\tan\theta d\theta = \frac{du}{\sec\theta} = \frac{du}{u}$. When $\theta = 0$, $u = \sec 0 = 1$, and when $\theta = \frac{\pi}{4}$, $u = \sec\frac{\pi}{4} = \sqrt{2}$.

$$\int_0^{\pi/4} \frac{\tan\theta}{\sqrt{\sec\theta}}\,d\theta = \int_1^{\sqrt{2}} \frac{1}{\sqrt{u}}\,\frac{du}{u} = \int_1^{\sqrt{2}} \frac{du}{u\sqrt{u}}$$

7. C If $u = \ln x$, then $du = \frac{1}{x}dx$. When $x = e$, $u = \ln e = 1$ and when $x = e^2$, $u = \ln e^2 = 2$.

$$\int_e^{e^2} \frac{1-(\ln x)^2}{x}\,dx = \int_1^2 (1-u^2)\,du$$

8. Let $u = x^3$, then $du = 3x^2dx$. When $x = 1$, $u = 1$ and when $x = 2$, $u = 8$.

$$\int_1^2 x^2 f(x^3)\,dx = \overbrace{\int_1^8 f(u)\,\frac{1}{3}du}^{x^2dx=\frac{1}{3}du} = \overbrace{\frac{1}{3}\int_1^8 f(x)\,dx}^{\text{Integrals do not depend on the variable that is used.}} = \frac{1}{3}(15) = 5$$